Introduction to general relativity

6.1 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold. We will define the *Riemann curvature tensor* $R : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ by

$$R(X,Y)Z \doteq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(a) Show that R is indeed a $(1,3)$-tensor field which is antisymmetric in its first two arguments.

Let V be a Killing vector field on $(\mathcal{M}, g)$ and $\gamma : (a,b) \to \mathcal{M}$ a geodesic for g.

(b) Prove that, for any vector field Z along γ:

$$g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma} V,\, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma)\big) = 0.$$

Deduce that the components of V^k of V in any local coordinate system around a given point on γ satisfy a second order linear ODE along γ.

(c) Show that if $V|_p = 0$ and $\nabla V|_p = 0$ for some $p \in \mathcal{M}$, then $V = 0$ on the whole connected component of $\mathcal{M}$ containing p.

(d) What is the maximum dimension of the Lie algebra of Killing fields on a connected Lorentzian manifold of dimension $n+1$? Compare it with the dimension of the Killing algebra on Minkowski spacetime.

Solution. (a) Showing that R is a tensor field amounts to showing that it is C^∞-multilinear in all its arguments. This is a simple calculation (using successively the fact that ∇ satisfies the Leibniz rule with respect to its second argument and the fact that $[fX, Y] = f[X,Y] - Y(f)X$). The antisymmetry of $R(X,Y)Z$ in X, Y also follows directly from the definition of R.

(b) Recall that a Killing vector field V satisfies the identity

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0 \tag{1}$$

for all vector fields X, Y. As a special case of (1),

$$g(\nabla_X V, X) = 0 \quad \text{for all vector fields } X. \tag{2}$$

Let $\gamma : (a,b) \to \mathcal{M}$ be a geodesic for g (i.e. satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$) and Z be any vector field along γ. The identity that we have to prove, namely

$$g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma} V, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big) = 0, \tag{3}$$

is a pointwise identity, i.e. in order to show that it is true at a point $p \in \gamma$ it suffices to know the vector fields $V, Z, \dot\gamma$ only in a small neighborhood of p. To this end, without loss of generality, we can assume (by restricting γ to a subdomain of (a,b)) that γ doesn't intersect itself and is contained in a single coordinate chart; in this way, we can extend $\dot\gamma$ and Z (in a non-unique way) to vector fields defined on the whole of $\mathcal{M}$ (using suitable cut-off functions in the given coordinate chart around p). In this way, we can make sense of expressions like $\nabla_Z \dot\gamma$ (even though the value of such an expression could depend on our precise choice of extending $\dot\gamma$ and Z beyond γ).

Applying (1) for $X = \dot\gamma$ and $Y = Z$ and obtain after differentiating along γ:

$$\begin{aligned}
0 &= \frac{d}{dt}\Big(g(\nabla_{\dot\gamma(t)}V|_{\gamma(t)}, Z|_{\gamma(t)}) + g(\nabla_Z V|_{\gamma(t)}, \dot\gamma(t))\Big)\\
&= \dot\gamma\Big(g(\nabla_{\dot\gamma}V, Z) + g(\nabla_Z V, \dot\gamma)\Big)\\
&= g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}Z\big) + g\big(\nabla_{\dot\gamma}\nabla_Z V, \dot\gamma\big) + g\big(\nabla_Z V, \nabla_{\dot\gamma}\dot\gamma\big)\\
&= g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}Z\big) + g\big(\nabla_{\dot\gamma}\nabla_Z V, \dot\gamma\big).
\end{aligned}$$

Using the definition of the Riemann curvature tensor, we can express

$$\nabla_{\dot\gamma}\nabla_Z V = R(\dot\gamma, Z)V + \nabla_Z\nabla_{\dot\gamma}V + \nabla_{[\dot\gamma,Z]}V.$$

Thus, returning to the above calculation, we have:

$$\begin{aligned}
0 \;&=\; g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big) + g\big(\nabla_Z\nabla_{\dot\gamma}V, \dot\gamma\big) + g\big(\nabla_{[\dot\gamma,Z]}, \dot\gamma\big)\\
&=\; g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big) + \Big(Z\big(g(\nabla_{\dot\gamma}V, \dot\gamma)\big) - g\big(\nabla_{\dot\gamma}V, \nabla_Z\dot\gamma\big)\Big) + g\big(\nabla_{[\dot\gamma,Z]}V, \dot\gamma\big)\\
&\overset{(2)\text{ for } X=\dot\gamma}{=}\; g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big)\\
&\qquad + g\big(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}Z - \nabla_Z\dot\gamma\big) + g\big(\nabla_{[\dot\gamma,Z]}V, \dot\gamma\big)\\
&\overset{\nabla\text{ is symmetric}}{=}\; g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big)\\
&\qquad + g\big(\nabla_{\dot\gamma}V, [\dot\gamma, Z]\big) + g\big(\nabla_{[\dot\gamma,Z]}V, \dot\gamma\big)\\
&\overset{(1)}{=}\; g\big(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V, Z\big) + g\big(R(\dot\gamma, Z)V, \dot\gamma\big),
\end{aligned}$$

i.e. (3) holds.

In any local coordinate chart $(x^0, \dots, x^n)$, choosing $Z = \frac{\partial}{\partial x^\alpha}$ in the identity (3), we have

$$0 = g_{\alpha\beta}(\nabla_{\dot\gamma}\nabla_{\dot\gamma}V)^\beta + R_{\kappa\lambda\mu\alpha}\dot\gamma^\mu\dot\gamma^\kappa V^\lambda \tag{4}$$

$$\begin{aligned}
&\Rightarrow 0 = (\nabla_{\dot\gamma}\nabla_{\dot\gamma}V)^\beta + g^{\alpha\beta}R_{\kappa\lambda\mu\alpha}\dot\gamma^\mu\dot\gamma^\kappa V^\lambda\\
&\qquad = \frac{d^2}{dt^2}V^\beta + \frac{d}{dt}\Big(\Gamma^\beta_{\mu\nu}\dot\gamma^\mu V^\nu\Big) + \Gamma^\beta_{\mu\nu}\dot\gamma^\mu\frac{dV^\nu}{dt} + \Gamma^\beta_{\mu\nu}\Gamma^\nu_{\kappa\lambda}\dot\gamma^\mu\dot\gamma^\kappa V^\lambda + g^{\alpha\beta}R_{\kappa\lambda\mu\alpha}\dot\gamma^\mu\dot\gamma^\kappa V^\lambda,
\end{aligned} \tag{5}$$

where we used the fact that, if Z is a vector field along γ, then $(\nabla_{\dot\gamma}Z)^\beta = \frac{d}{dt}Z^\beta + \Gamma^\beta_{\mu\nu}\dot\gamma^\mu Z^\nu$. Notice that the above is a *linear* system of second order ODEs for the components $V^\beta(t)$ of $V|_{\gamma(t)}$.

(c) Let $p \in \mathcal{M}$ be as in the statement of the exercise, i.e. $V|_p = 0$ and $\nabla V|_p = 0$ (the latter equality means that $\nabla_X V|_p = 0$ for any vector field X). Let $\mathcal{Z} \subset \mathcal{M}$ be the subset of $\mathcal{M}$ where V and ∇V vanish, i.e.

$$\mathcal{Z} = \big\{q \in \mathcal{M} : V|_q = 0 \text{ and } \nabla V|_q = 0\big\}.$$

We want to show that $\mathcal{Z} = \mathcal{M}$. Since $\mathcal{M}$ is connected, it suffices to show that $\mathcal{Z}$ is both open and closed in $\mathcal{M}$, and that it is non-empty. Since $p \in \mathcal{Z}$, we know that $\mathcal{Z} \neq \emptyset$. Since V is a continuous (in fact, smooth) vector field, we also know that $\mathcal{Z}$ has to be closed. Therefore, it only remains to show that $\mathcal{Z}$ is open.

Let $q \in \mathcal{Z}$. We will show that there exists an open neighborhood $\mathcal{U}$ of q in $\mathcal{M}$ such that $V = 0$ on $\mathcal{U}$; this will imply that $\mathcal{U} \subset \mathcal{Z}$ and, thus, that $\mathcal{Z}$ is open. Let $\gamma : (-a, a) \to \mathcal{M}$ be any geodesic of $(\mathcal{M}, g)$ with $\gamma(0) = q$ and let $(x^0, \dots, x^n)$ be any coordinate system on a neighborhood $\mathcal{W}$ around q. The condition $q \in \mathcal{Z}$ implies that

$$V^\beta|_{\gamma(0)} = 0 = \partial_\alpha V^\beta|_{\gamma(0)}. \tag{6}$$

Therefore, the components $V^\beta(t)$ of $V|_{\gamma(t)}$ satisfy the second order ODE (5) $V^\beta(0) = \frac{dV^\beta}{dt}(0) = 0$; by the uniqueness of solutions to systems of linear ODEs, we infer that $V \equiv 0$ on $\gamma \cap \mathcal{W}$. As a result, $V \equiv 0$ on the subset of $\mathcal{W}$ covered by geodesics passing through q, i.e. by the image of the exponential map $\exp_q : \Omega_q \subset T_q\mathcal{M} \to \mathcal{M}$. Since $\exp_q$ is a local diffeomorphism (because $d\exp_q|_0 = \mathrm{Id}$), we know that the image of $\exp_q$ contains an open neighborhood of q; thus, $\mathcal{Z}$ contains an open neighborhood of q.

(d) As a consequence of part (c) of this exercise, we know that, on a connected Lorentzian manifold $(\mathcal{M}, g)$, if two Killing fields $V^{(1)}$, $V^{(2)}$ satisfy $V^{(1)}|_p = V^{(2)}|_p$ and $\nabla V^{(1)}|_p = \nabla V^{(2)}|_p$ at a point $p \in \mathcal{M}$, then $V^{(1)} = V^{(2)}$ on the whole of $\mathcal{M}$. Therefore, the maximum number of linearly independent Killing vector fields on $\mathcal{M}$ is at most as large as the number of independent components of the tangent vector $V|_p$ and the tensor $\nabla V|_p$. Note that not all components of $\nabla V|_p$ are independent: In any local coordinate system around p, the relation (1) can be written as

$$\nabla_\alpha V_\beta + \nabla_\beta V_\alpha = 0.$$

Therefore, $\nabla V|_p$ has at most as many independent components as an antisymmetric matrix. Collecting these observations, if $\dim\mathcal{M} = n+1$, then the number of independent components of $V|_p$ and $\nabla V|_p$ (and, therefore, the maximum dimension of the Killing algebra of $(\mathcal{M}, g)$) is $(n+1) + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$. Note that this upper bound is optimal, since it is achieved in the case of Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$ and de-Sitter spacetime $(\mathbb{R} \times \mathbb{S}^n, g_{dS})$.

6.2 (a) Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let V be a Killing vector field on $(\mathcal{M}, g)$. Show that for any geodesic $\gamma : I \to \mathcal{M}$, the inner product $g(\dot\gamma, V)$ is constant along V.

*(b) Let $\mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}}$ be a product manifold equipped with a Lorentzian metric g of the form

$$g = -f \cdot dt \otimes dt + dt \otimes \omega + \omega \otimes dt + \bar{g},$$

where

- $t : \mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}} \to \mathbb{R}$ is the projection on the first factor.
- f, h are smooth functions on $\bar{\mathcal{M}}$.
- ω is an 1-form on $\bar{\mathcal{M}}$.
- $\bar{g}$ is a Riemannian metric on $\bar{\mathcal{M}}$.

Let $\mathfrak{E} \subset \bar{\mathcal{M}}$ be the set where $f < 0$. Show that, for every $p \in \mathbb{R} \times \mathfrak{E}$, there exists a maximally extended **null** geodesic $\gamma : (a, b) \to \mathcal{M}$ for g with $\gamma(0) = p$ which does not escape $\mathbb{R} \times \mathfrak{E}$ (i.e. $\gamma(s) \in \mathbb{R} \times \mathfrak{E}$ for all $s \in (a, b)$).

Solution. (a) Using the identity (2) for a Killing vector field V for $X = \dot{\gamma}$, we can readily compute along a geodesic $\dot{\gamma}$ of $(\mathcal{M}, g)$:

$$\frac{d}{dt}\big(g(V,\dot{\gamma})\big) = \dot{\gamma}\big(g(V,\dot{\gamma})\big) = g\big(\nabla_{\dot{\gamma}}V, \dot{\gamma}\big) + g\big(V, \nabla_{\dot{\gamma}}\dot{\gamma}\big) = 0 + 0.$$

Therefore, $g(V,\dot{\gamma})$ is constant along γ.

(b) Let T be the vector field which generates the translations of $\mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}}$ in the $\mathbb{R}$ factor, fixed by the requirement that $T(t) = 1$ (where $t : \mathcal{M} == \mathbb{R} \times \bar{\mathcal{M}} \to \mathbb{R}$ is the projection on the first factor). If $(x^1, \dots, x^n)$ is a local coordinate system on a subset $\mathcal{U}$ of $\bar{\mathcal{M}}$, we can extend it to a local coordinate system $(t, x^1, \dots, x^n)$ on $\mathbb{R} \times \mathcal{U} \subset \mathcal{M}$; in any such coordinate system, T is simply the vector field $\frac{\partial}{\partial t}$. Moreover, in such a coordinate system the metric g takes the form

$$g = -f dt^2 + 2\omega_i dt dx^i + \bar{g}_{ij} dx^i dx^j,$$

where ω_i and $\bar{g}_{ij}$ depend only on $x^1, \dots, x^n$; therefore, $T = \frac{\partial}{\partial t}$ is a *Killing* vector field on $(\mathcal{M}, g)$. Moreover, in view of the form of g, we have

$$g(T,T) = -f. \tag{7}$$

Furthermore, for any $p = (t_0, q) \in \mathcal{M} = \mathbb{R} \times \bar{\mathcal{M}}$, if $v \in T_p\mathcal{M}$ satisfies $dt(v) = 0$ (i.e. v is tangential to the slice $\{t_0\} \times \bar{\mathcal{M}}$ at p), then $g(v,v) = \bar{g}_{ij} v^i v^j$. Since $\bar{g}$ is Riemannian, we infer that any such v is *spacelike*. Equivalently,

$$dt(v) \neq 0 \quad \text{if } g(v,v) \leqslant 0 \text{ and } v \neq 0. \tag{8}$$

If $\mathfrak{E} = \{q \in \bar{\mathcal{M}} : f(q) < 0\}$ is non-empty and $p \in \mathbb{R} \times \mathfrak{E}$, then (7) implies that T is spacelike at p. Therefore, there exists a tangent vector $v \in T_p\mathcal{M} \setminus 0$ such that v is null (i.e. $g(v,v) = 0$) and satifies $g(v, T|_p) > 0$ and $dt(v)|_p > 0$ (note that any null vector v cannot satisfy $dt(v) = 0$, since that would imply that v is *spacelike* by (8)). Note that this statement is the same as saying that in any Lorentzian inner product space (V, m), if $v_1 \neq 0$ is a spacelike vector then there exists a null vector v_2 with $m(v_1, v_2) > 0$ such that v_2 is future directed (it is an easy exercise to show that this is true).

Let $\gamma : (a,b) \to \mathcal{M}$ (with $0 \in (a,b)$) be the *maximally extended* geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Note the following facts:

- Since γ is a geodesic, $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ; therefore, $g(\dot{\gamma}(s), \dot{\gamma}(s)) = g(v,v) = 0$ for all $s \in (a,b)$, i.e. γ is a null geodesic.
- By part (a) of this exercise, $g(\dot{\gamma}, T)$ stays constant along γ; therefore, $g(\dot{\gamma}(s), T|_{\gamma(s)}) > 0$ for all $s \in (a,b)$.
- Since $\dot{\gamma}$ is nowhere spacelike along γ, by (8) we have that $dt(\dot{\gamma}(s)) \neq 0$ for $s \in (a,b)$; therefore, $dt(\dot{\gamma}(s)) > 0$ (since this is the case for $s = 0$).

The above facts now imply that there can be no $s \in (a,b)$ such that $\gamma(s) \in \mathcal{M} \setminus \big(\mathbb{R} \times \mathfrak{E}\big)$: Assuming, for the sake of contradiction, that such an s exists, then $f \geqslant 0$ at $\gamma(s)$, i.e. $T|_{\gamma(s)}$ is null or timelike. Since $dt(\dot{\gamma}(s)) > 0$ and $dt(T) = T(t) = 1 > 0$, $T|_{\gamma(s)}$ and $\dot{\gamma}(s)$ belong to the same timecone in

$(T_{\gamma(s)}\mathcal{M}, g)$; the inner product of two causal vectors belonging to the same timecone is always $\leqslant 0$ (it is easy to check that), which is a *contradiction* in view of the fact that $g(\dot{\gamma}(s), T|_{\gamma(s)}) > 0$ for all $s \in (a, b)$.

6.3 Consider the manifold $\mathcal{M} = \mathbb{R} \times \mathbb{R}$ equipped with the Lorentzian metric

$$g_{AdS} = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2$$

(this is known as the $1+1$ dimensional *Anti-de Sitter metric*). Note that $(\mathcal{M}, g)$ can be thought of as the universal Lorentzian cover of the (topological) cylinder $\mathcal{S} = \{-x^2 - y^2 + r^2 = +1\}$ in the pseudo-Euclidean space $(\mathbb{R}^{2+1}, \eta_{(2,1)})$, where $\eta_{(2,1)} = -dx^2 - dy^2 + dr^2$.

(a) Show that $(\mathcal{M}, g_{AdS})$ is timelike geodesically complete, i.e. that all timelike geodesics of g_{AdS} can be extended on the whole of $\mathbb{R}$.

(b) Show that *all* timelike geodesics γ passing through $(t, r) = (0, 0)$ also pass through $(t, r) = (k\pi, 0)$, $k \in \mathbb{Z}$. Show that there exists a point $p \in \mathcal{I}^+[(0,0)]$ such that $(0, 0)$ and p cannot be connected with a timelike geodesic.

Solution. (a) Let $\gamma : (a, b) \to \mathcal{M}$, $s \to \gamma(s) = \big(t(s), r(s)\big)$, be a *timelike* geodesic; assume (by shifting the parametrization, if necessary) that $0 \in (a, b)$. Instead of using the second order geodesic equation, we will find a closed expression for γ using first order *conserved quantities* along γ.

- Since γ is geodesic, the quantity $g_{AdS}(\dot{\gamma}, \dot{\gamma})$ is constant along γ; since we assumed that γ is timelike, $g_{AdS}(\dot{\gamma}, \dot{\gamma})$. Expressed in coordinates, we have

$$-\big(1 + r(s)^2\big)\dot{t}^2(s) + \frac{1}{1 + r(s)^2}\dot{r}^2(s) = -A, \quad A > 0. \tag{9}$$

- Since the components of g_{AdS} in the (t, r) coordinate system are independent of t, the vector field $T = \frac{\partial}{\partial t}$ is a Killing vector field for g_{AdS} and, thus, $g_{AdS}(\dot{\gamma}, T)$ is constant along γ. Therefore,

$$-\big(1 + r(s)^2\big)\dot{t}(s) = -B, \quad B \in \mathbb{R}. \tag{10}$$

  Notice that, plugging the expression for $\dot{t}$ from (10) in (9), we infer that

$$B^2 - A = r(s)^2 A + \dot{r}(s)^2 \geqslant 0. \tag{11}$$

Combining (9) and (10), we obtain:

$$\begin{aligned} \frac{dt}{ds} &= \frac{B}{1 + r(s)^2}, \\ \frac{dr}{ds} &= \pm\sqrt{B^2 - \big(1 + r(s)^2\big)A}. \end{aligned} \tag{12}$$

Note that the equation for $r(s)$ can be explicitly solved:

- In the case when $B^2 - A > 0$,

  $$\int_{r(0)}^{r(s)} \frac{dr}{\sqrt{(B^2-A)-Ar^2}} = \pm s \quad \Rightarrow \quad \arcsin\Big(\sqrt{\frac{A}{B^2-A}}r(s)\Big) - \arcsin\Big(\sqrt{\frac{A}{B^2-A}}r(0)\Big) = \pm\sqrt{A}s.$$

  Therefore,

  $$r(s) = \sqrt{\frac{B^2-A}{A}}\sin\Big(\pm\sqrt{A}s + \rho_0\Big) \quad \text{for } \rho_0 = \arcsin\Big(\sqrt{\frac{A}{B^2-A}}r(0)\Big).$$

- In the case when $B^2 - A = 0$, (11) implies that

  $$r(s) = 0$$

  (this is only the case if $r(0) = 0$ and $\dot{r}(0) = 0$).

Notice also that, given the closed form for $r(s)$, the function $t(s)$ can be also determined by integrating the equation for $\frac{dt}{ds}$, namely

$$t(s) = t(0) + \int_0^s \frac{B}{1 + r(s)^2}\, ds.$$

We therefore note that the expression for $\gamma(s) = (t(s), r(s))$ can be extended for $s \in (-\infty, +\infty)$ independently of the choice of the initial point $(t(0), r(0))$. In other words, every maximal timelike geodesic in $(\mathcal{M}, g_{AdS})$ is defined for $s \in \mathbb{R}$ and, thus, $(\mathcal{M}, g_{AdS})$ is timelike geodesically complete.

(b) Let $\gamma : \mathbb{R} \to \mathcal{M}$, $s \to (t(s), r(s))$ be a timelike geodesic such that $\gamma(0) = (0, 0)$. It will be more convenient to change the parametrization of γ, so that it is parametrized by the coordinate t rather than the natural parameter s (this is possible since $\dot{t} \neq 0$, according to (12) and the fact that $B^2 \geqslant A > 0$; note that, with this parametrization, γ will not be a geodesic. Moreover, in view of the fact that, along γ, we have (see (11)):

$$B^2 - A = r(s)^2 A + \dot{r}(s)^2 \geqslant r(s)^2 A,$$

we know that

$$\sup_{s\in\mathbb{R}} r(s)^2 \leqslant \frac{B^2-A}{A}.$$

Thus, from the equation

$$\frac{dt}{ds} = \frac{B}{1 + r(s)^2}$$

we infer that $\inf_{s\in\mathbb{R}} \left|\frac{dt}{ds}\right| \geqslant \frac{A}{|B|} > 0$ and, therefore,

$$\lim_{s\to+\infty} t(s) = \begin{cases} +\infty, & B > 0, \\ -\infty, & B < 0 \end{cases}$$

(and similarly for $s \to -\infty$).

In particular, we can calculate:

$$\Big(\frac{dr}{dt}\Big)^2=\Big(\frac{\frac{dr}{ds}}{\frac{dt}{ds}}\Big)^2=(1+r^2)^2\Big(\frac{B^2-A}{B^2}-\frac{A}{B^2}r^2\Big)$$

Depending on the value of B^2-A, the above equation admits two types of solutions:

- If $B^2-A=0$, then, as explained above, $r(t)=0$.
- If $B^2-A>0$, then, after integrating the above equation, we get

$$\int_0^{r(t)}\frac{dr}{(1+r^2)\sqrt{\frac{B^2-A}{B^2}-\frac{A}{B^2}r^2}}=\pm t\quad\Rightarrow\quad\frac{r(t)}{\sqrt{\frac{B^2-A}{B^2}-\frac{A}{B^2}(r(t))^2}}=\pm\tan t.$$

Therefore, we infer that, for any $A>0$ and $B^2\geqslant A$, the point $(t,r)=(k\pi,0)$, $k\in\mathbb{Z}$, belongs to the curve γ. Thus, every maximal timelike geodesic through $(0,0)$ passes through $(\pi,0)$.

For $\epsilon>0$ sufficiently small, let us consider the points $p=(0,0)$ and $q=(\pi,\epsilon)$. From the explicit formula for timelike geodesics through p that we obtained above, it is easy to see that every such geodesic $s\to(t(s),r(s))$ satisfies $r(s)=0$ when $t(s)=\pi$. Therefore, no such geodesic passes through the point q. On the other hand, there is a future directed timelike curve connecting p to q, namely the curve $s\to(t(s),r(s))=(s,\frac{\epsilon}{\pi}s)$, $s\in[0,\pi]$. Thus, p and q are connected by a timelike curve but not with a timelike geodesic.

Alternative approach: There is a more "geometric" way of constructing the geodesics of Anti-de Sitter spacetime. The starting point for this is the following fact: On any pseudo-Euclidean space $(\mathbb{R}^m,\eta_{(p,q)})$, where $\eta_{(p,q)}$ is the inner product of signature (p,q), $p+q=m$, given by the matrix

$$\eta_{(p,q)}=\mathrm{diag}(\underbrace{-1,\dots,-1}_{\sharp\, p},\underbrace{+1,\dots,+1}_{\sharp\, q}),$$

let $\mathcal{S}\subset\mathbb{R}^m$ be a connected component of the set $\big\{x\in\mathbb{R}^m:\quad\eta_{(p,q)}(x,x)=M\big\}$ for some $M\in\mathbb{R}\setminus\{0\}$ and let g denote the induced pseudo-Riemannian metric on $\mathcal{S}$. Note that the standard round sphere in Euclidean space, de-Sitter spacetime (viewed as a subset of $(\mathbb{R}^{n+1},\eta_{(1,n)})$) and anti-de Sitter spacetime after the identification $t\equiv t+2\pi$ (see the statement of this exercise) can all be expressed like this. In this case, the intersection of any 2-plane Π passing through 0 with $\mathcal{S}$ is (up to reparametrization) a *geodesic* of $(\mathcal{S},g)$ (and, in fact, all geodesics of $\mathcal{S}$ can be expressed in this way). This statement should be familiar to you in the case of the round sphere, but the proof of that statement actually is independent of the signature of the ambient inner product space, and consists of the following observations:

- The acceleration of any curve lying in a 2-plane Π is tangential to Π.
- At any point $x\in\mathcal{S}=\big\{x\in\mathbb{R}^m:\quad\eta_{(p,q)}(x,x)=M\big\}$, the normal vector $\hat{n}$ with respect to $\eta_{(p,q)}$ is parallel to x (i.e. to the radial vector). Moreover, if $M\neq 0$, the normal vector $\hat{n}$ is not tangent to $\mathcal{S}$ (in particular, the induced metric g is a non-degenerate pseudo-Riemannian metric).

- As a consequence of the above, if a curve $\gamma : (a,b) \to \mathcal{S}$ parametrizes the intersection $\Pi \cap \mathcal{S}$ for some 2-plane Π and $x \in \gamma$, then $T_x\Pi$ is spanned by $\hat{n} \parallel x$ and $\dot{\gamma}$, so the acceleration $\ddot{\gamma}$ (viewed as a curve in $\mathbb{R}^m$) lies in the span of $\hat{n}$ and $\dot{\gamma}$.
- For any submanifold $\mathcal{M}^{m-1} \subset \mathbb{R}^m$ with induced non-degenerate pseudo-Riemannian metric $\bar{g}$, the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ is obtained as the orthogonal projection on $T\mathcal{M} \subset T\mathbb{R}^m$ of the flat connection of $(\mathbb{R}^m, \eta_{(p,q)})$. In particular, a curve γ in $(\mathcal{M}, \tilde{g})$ is a geodesic if its acceleration $\ddot{\gamma}$ (when viewed as a curve in $\mathbb{R}^m$) is parallel to the normal $\hat{n}$ of $\mathcal{M}$. Moreover, γ can be reparametrized to be a geodesic of $(\mathcal{M}, \tilde{g})$ if $\ddot{\gamma}$ lies in the span of $\hat{n}$ and $\dot{\gamma}$.

As a consequence of the above statements, we deduce that the geodesics of the cylinder $\mathcal{S} = \{-x^2 - y^2 + r^2 = +1\}$ (viewed as a subset of $(\mathbb{R}^{2+1}, \eta_{(2,1)})$) are the intersections of $\mathcal{S}$ with 2-planes of $\mathbb{R}^{2+1}$ passing through the origin. In particular, timelike geodesics of $\mathcal{S}$ correspond to the intersection with timelike planes, which can be easily seen to be ellipses wrapping around $\mathcal{S}$. Note that, since both $\mathcal{S}$ and any such timelike plane Π are symmetric with respect to the origin, if $x \in \Pi \cap \mathcal{S}$ then we also have $-x \in \pi \cap \mathcal{S}$. In particular, every timelike geodesic through x will have to also pass through $-x$. Unwrapping the cylinder $\mathcal{S}$ along the $\mathbb{S}^1$ direction, it is easy to verify that this observation corresponds to the statement of part (b) of this exercise.

6.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $S \subset \mathcal{M}$ be a smooth *null* hypersurface. Let L be a non-zero vector field along S such that, for every $p \in S$, $L|_p \perp T_pS$.

(a) Show that L is tangent to S (i.e. $L|_p \in T_pS$ for all $p \in S$).

(b) Show that, for any two vector fields X, Y tangent to S, we have

$$g\big(\nabla_X L, Y\big) = g\big(\nabla_Y L, X\big).$$

(*Hint: You might want to use the fact that, if V, W are two vector fields tangent to a submanifold $\mathcal{N} \subset \mathcal{M}$, then $[V, W]$ is also tangent to $\mathcal{N}$.*)

(c) Using the above formula, show that $\nabla_L L|_p \perp T_pS$ for all $p \in S$. Show that

$$\nabla_L L = \kappa L \quad \text{for some } \kappa : S \to \mathbb{R}.$$

Deduce that the integral curves of L in S are (up to reparametrization) null geodesics. Infer that any null hypersurface of $(\mathbb{R}^{1+n}, \eta)$ is a union of null lines.

(d) Assume, in addition, that there exists a Killing vector field V on $(\mathcal{M}, g)$ such that V is collinear with L along S (such a null hypersurface is called a *Killing horizon*). By replacing L with V in the above arguments, we infer that

$$\nabla_V V|_S = \kappa V|_S.$$

Show that the function κ above is a *constant* along the null generators of S, i.e. $V(\kappa) = 0$. This is the *surface gravity* of the Killing horizon.

(e) Show that the hyperplane $H = \{x = t\}$ in $(\mathbb{R}^{3+1}, \eta)$ (with the usual Cartesian coordinate system (t, x, y, z)) is a Killing horizon (and find the corresponding Killing vector field of $(\mathbb{R}^{3+1}, \eta)$. Can you compute its surface gravity?

Solution. (a) For any $p \in S$, let us consider the Lorentzian inner product space $(T_p\mathcal{M}, g|_p)$ and let $V = T_pS \subset T_p\mathcal{M}$. Our assumption that S is a null hypersurface of $\mathcal{M}$ implies that V is a null codimension 1 hyperplane of $T_p\mathcal{M}$, i.e. the form $g|_p$ restricted to V is *degenerate*: There exists a $v \in V$ such that $g|_p(v, w) = 0$ for *all* $w \in V$. Therefore, $V \subset v^\perp = \{w \in T_p\mathcal{M} : g|_p(v, w) = 0\}$. Since V is of codimension 1 and $v^\perp$ is also of codimension 1 (in view of the fact that $g|_p$ is non-degenerate on $T_p\mathcal{M}$, we cannot have $g(v, w) = 0$ for all $w \in T_p\mathcal{M}$), we have

$$v^\perp = V$$

In particular, for any $w \notin V$, the above relation implies that, necessarily, $g|_p(v, w) \neq 0$. Since we assumed that$L|_p \perp V$, we must have $g|_p(v, L) = 0$; thus, $L|_p \in V$. Note that, in fact, $L|_p$ must be parallel to v: Since $L|_p \in V$ and $L|_p \perp V$, we have $L|_p \perp L|_p \Leftrightarrow g|_p(L, L) = 0$; but as we showed in Exercise 2.1, if two null vectors (in this case v and $L|_p$ are perpendicular to each other, then they have to be parallel).

Remark. This is a general statement about Lorentzian inner product spaces (W, m): The perpendicular direction to a null hyperplane V lies inside V, and it is simply the direction of the null generator of V (namely the direction in which $m|_V$ degenerates).

(b) We can readily calculate at any point on S:

$$g(\nabla_X L, Y) = X\big(g(L, Y)\big) - g(L, \nabla_X Y) = -g(L, \nabla_X Y)$$

(since $g(L, Y) = 0$ on S and we are differentiating in the direction of X which is tangential to S) and, similarly (with the roles of X and Y inverted):

$$g(\nabla_Y L, X) = -g(L, \nabla_Y X).$$

Subtracting the above expressions and using the fact that the Levi-Civita connection is torsion-free, we obtain:

$$g(\nabla_X L, Y) - g(\nabla_Y L, X) = -g(L, \nabla_X Y) + g(L, \nabla_Y X) = g(L, \nabla_Y X - \nabla_X Y) = g(L, [X, Y]) = 0,$$

since $[X, Y]$ is tangential to S (given that X, Y are; you can check this in local coordinates in which S is the level set $\{x^0 = 0\}$).

(c) For any $p \in S$, we have showed in part (a) that $L|_p \in T_pS$; therefore, plugging in $X = L$ in the relation proven in part (b), we obtain that, for any $p \in S$ and any $Y \in T_pS$:

$$g|_p(\nabla_L L, Y) = g|_p(\nabla_Y L, L) = \frac{1}{2} Y\big(g(L, L)\big)\big|_p = 0,$$

since $g(L, L) = 0$ on S (recall that $L \perp T_pS$ and $L \in T_pS$) and Y is a direction tangential to S. Therefore,

$$\nabla_L L\big|_p \perp T_pS \overset{L \in T_pS}{\Rightarrow} \nabla_L L\big|_p \perp L|_p.$$

Since any two perpendicular null vectors have to be parallel to each other (see Ex. 2.1), we infer that

$$\nabla_L L\big|_p = \kappa_p L|_p \quad \text{for some } \kappa_p \in \mathbb{R}.$$

Thus, along S, we have

$$\nabla_L L = \kappa L \quad \text{for some } \kappa : S \to \mathbb{R}$$

(note that κ has to be a smooth function, since both $\nabla_L L$ and L are smooth vector fields and L doesn't vanish).

Let $\gamma : [0,l] \to \mathcal{M}$ be an integral curve of L, i.e. $\gamma(0) \in S$ and $\dot{\gamma}(t) = L|_{\gamma(t)}$ for al $t \in [0,l]$; note that such an integral curve has to lie inside S, since L is tangnetial to S (again, this can be easily verified in local coordinates where S is the level set of one of the coordinate functions). Then $\dot{\gamma}$ satisfies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa|_{\gamma(t)}\dot{\gamma}. \tag{13}$$

This is the equation of a reparametrized geodesic: If we reparametrize γ as

$$\tilde{\gamma}(s) = \gamma \circ h(s) \quad \text{for some strictly monotonic function } h : [0,l] \to [a,b],$$

then

$$\dot{\tilde{\gamma}}(s) = h'(s)\dot{\gamma}(h(s)),$$

and (noting that, if f is a function defined along $\tilde{\gamma}$, then $\frac{d}{ds}(f|_{\tilde{\gamma}(s)}) = \dot{\tilde{\gamma}}(f)|_{\tilde{\gamma}(s)}$

$$\begin{aligned}\nabla_{\dot{\tilde{\gamma}}(s)}\dot{\tilde{\gamma}}(s) &= \nabla_{\dot{\tilde{\gamma}}(s)}\big(h'(s)\dot{\gamma}(h(s))\big)\\ &= \dot{\tilde{\gamma}}(s)(h')\dot{\gamma}(h(s)) + h'(s)\nabla_{\dot{\tilde{\gamma}}(s)}\dot{\gamma}(h(s))\\ &= h''(s)\dot{\gamma}(h(s)) + h'(s)\nabla_{\dot{\gamma}(h(s))}\dot{\gamma}(h(s)).\end{aligned}$$

Using the relation (13), we therefore infer that

$$\nabla_{\dot{\tilde{\gamma}}(s)}\dot{\tilde{\gamma}}(s) = h''(s)\dot{\gamma}(h(s)) + \kappa|_{\gamma(h(s))}\dot{\gamma}(h(s)).$$

Choosing the function h so that $h''(s) = -\kappa|_{\gamma(h(s))}$, we therefore infer that $\nabla_{\dot{\tilde{\gamma}}(s)}\dot{\tilde{\gamma}}(s) = 0$, i.e. the reparametrization $\tilde{\gamma}$ of γ is a geodesic.

(d) Let V be a Killing vector field as in the statement of the Exercise. For any $p \in S$, let $\mathcal{U}$ be an open neighborhood of p in $\mathcal{M}$ with compact closure; for any $t > 0$ sufficiently small, let $\Phi^{(V)}_t : \mathcal{U} \to \mathcal{M}$ be the flow map associated to V (i.e. the flow along the integral lines of V for time t). Because V is tangent to S, we have that $\Phi^{(V)}_t(\mathcal{U} \cap S) \subset \mathcal{S}$. Moreover, because V is Killing, the map $\Phi^{(V)}_t : (\mathcal{U}, g) \to \big(\Phi^{(V)}_t(\mathcal{U}), g\big) \subset (\mathcal{M}, g)$ is an *isometry* (i.e. $\big(\Phi^{(V)}_t\big)_* g = g$). Our aim is to show that the isometric flow of V should leave the "geometric" relation

$$\nabla_V V|_S = \kappa V|_S$$

invariant, which should imply that κ is constant along the flow of V.

In general, if $\Phi : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ is an isometry between to Lorentzian (or, more generally, pseudo-Riemannian) manifolds, then Φ "preserves" any natural construction obtained from the metric tensor, such as the Levi-Civita connection. In particular, for any $X, Y \in \Gamma(\mathcal{N}_1)$:

$$\Phi^*\big(\nabla^{(1)}_X Y\big) = \nabla^{(2)}_{\Phi^* X}\Phi^* Y,$$

where $\nabla^{(i)}$ is the Levi-Civita connection of g_i; this can be seen by recalling that the Levi-Civita connection ∇ of a metric g is uniqueley determined by the formula of Koszul (for any three vector fields X, Y, Z):

$$2g\big(\nabla_X Y, Z\big) = X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) - g\big(X,[Y,Z]\big) + g\big(Z,[X,Y]\big) + g\big(Y,[Z,X]\big).$$

In our case, since $\Phi^{(V)}_t$ is an isometry, we must have

$$\big(\Phi^{(V)}_t\big)^*\big(\nabla_V V\big) = \nabla_{\left(\Phi^{(V)}_t\right)^* V}\Big(\big(\Phi^{(V)}_t\big)^* V\Big). \tag{14}$$

Note that the map $\big(\Phi^{(V)}_t\big)^*_p : T_p\mathcal{M} \to T_{\Phi^{(V)}_t(p)}\mathcal{M}$ satisfies

$$\big(\Phi^{(V)}_t\big)^*_p(V|_p) = V|_{\Phi^{(V)}_t(p)} \tag{15}$$

(this is in general true for the flow map of any vector field; you can readily verify that in a system of coordinates where $V = \frac{\partial}{\partial x^1}$ and $\Phi^{(V)}_t$ is just a coordinate translation $x^1 \to x^1 + t$). Therefore, using the given relation

$$\nabla_V V|_S = \kappa V|_S$$

in (14), we infer that

$$\big(\Phi^{(V)}_t\big)^*_p\big(\kappa_p V|_p\big) = \kappa_{\Phi^{(V)}_t(p)} V|_{\Phi^{(V)}_t(p)}$$

which, in view of (15) (and the linearity of $\big(\Phi^{(V)}_t\big)^*_p$), implies that

$$\kappa_p V|_{\Phi^{(V)}_t(p)} = \kappa_{\Phi^{(V)}_t(p)} V|_{\Phi^{(V)}_t(p)} \Rightarrow \kappa_p = \kappa_{\Phi^{(V)}_t(p)},$$

i.e. κ is constant along the flow of p.

(e) Let $\mathcal{H} = \{x = t\} \subset (\mathbb{R}^{3+1}, \eta)$. It is easy to verify that $\mathcal{H}$ is a null hypersurface with null generator $L = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$. Note also that the boost vector field

$$V = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$$

is a Killing vector field of $(\mathbb{R}^{3+1}, \eta)$ which is collinear to L along $\mathcal{H}$. In particular, $\mathcal{H}$ is a *Killing horizon.* Note also that

$$\nabla_V V = \nabla_{t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}}\big(t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\big) = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$$

and, therefore, along $\mathcal{H} = \{x = t\}$:

$$\nabla_V V|_{\mathcal{H}} = V|_{\mathcal{H}}.$$

In particular, $\kappa = 1$.

Remark. Note that V vanishes along the 2-surface $\mathcal{H} \cap \{t = 0\}$. It is an easy exercise to verify that, for any Killing horizon with *positive* surface gravity on which the null generators are complete geodesics, the Killing vector field must vanish at some point along the null generator towards the past in finite affine time. Another Killing horizon that we will encounter with a similar property will be the event horizon of the maximally extended Schwarzschild spacetime.